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2006 J. Phys. A: Math. Gen. 39 6987

(http://iopscience.iop.org/0305-4470/39/22/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6987–7001 doi:10.1088/0305-4470/39/22/013

SUSY approach to Pauli Hamiltonians with an axial symmetry

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Received 23 February 2006
Published 16 May 2006
Online at stacks.iop.org/JPhysA/39/6987

Abstract

A two-dimensional Pauli Hamiltonian describing the interaction of a neutral spin-$1/2$ particle with a magnetic field having axial and second-order symmetries is considered. After separation of variables, the one-dimensional matrix Hamiltonian is analysed from the point of view of supersymmetric quantum mechanics. Attention is paid to the discrete symmetries of the Hamiltonian and also to the Hamiltonian hierarchies generated by intertwining operators. The spectrum is studied by means of the associated matrix shape invariance. The relation between the intertwining operators and the second-order symmetries is established, and the full set of ladder operators that complete the dynamical algebra is constructed.

PACS numbers: 11.30.Pb, 03.65.Ge, 03.65.Fd, 02.30.Gp

1. Introduction

In this work we will study a Pauli Hamiltonian describing the interaction of a neutral spin-$1/2$ particle interacting with a magnetic field generated by an electric current-carrying straight wire. This system was introduced in [1], where it was analysed in the momentum space. Here, we will carry out a systematic study in the configuration space based on the techniques of supersymmetric (SUSY) quantum mechanics [2, 3], sometimes referred to as the factorization method [4], a technique that has already been used to study spin-$1/2$ Pauli equations (for different approaches, see for example [5]).

To motivate the specific form of the Pauli Hamiltonian for the present work, we will start with the formulation of its symmetry properties. We will see later that this additional

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information is also closely related to the factorization method. Thus, let us consider the Pauli Hamiltonian in the three-dimensional space

$$\mathcal{H}_3 = \frac{\boldsymbol{p}^2}{2m} + \mu\boldsymbol{\sigma}\cdot\boldsymbol{B}(\boldsymbol{x}) + V(\boldsymbol{x}), \tag{1.1}$$

where $\boldsymbol{B}(\boldsymbol{x})$ is a magnetic field, μ is the magnetic moment of the particle, $V(\boldsymbol{x})$ is a scalar potential and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_j being the Pauli matrices, that is,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

We take the usual notation for the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ and the momentum operators $p_k = -\mathrm{i}\hbar\partial/\partial x_k \equiv \partial x_k$. Now, we will impose some symmetries on this Hamiltonian in order to obtain the precise example we want to consider in this work.

1.1. Rotational–translational symmetry along the x_3-axis

We assume that the current-carrying wire is placed on the x_3-axis, and we look for the systems (1.1) allowing for first-order symmetries of the form

$$\mathcal{P}_3 = p_3 + a(\boldsymbol{x}) \qquad \mathcal{J}_3 = x_1 p_2 - x_2 p_1 + b(\boldsymbol{x}) \equiv L_3 + b(\boldsymbol{x}), \tag{1.3}$$

where $a(\boldsymbol{x})$ and $b(\boldsymbol{x})$ are Hermitian matrix-valued functions to be determined. By imposing that $\mathcal{J}_3$ and $\mathcal{P}_3$ are symmetries of $\mathcal{H}_3$, that is, $[\mathcal{J}_3, \mathcal{H}_3] = [\mathcal{P}_3, \mathcal{H}_3] = 0$, we are led to the following explicit expressions (up to an equivalence):

$$\mathcal{J}_3 = -\mathrm{i}\hbar\partial_\theta + \beta\sigma_3 \qquad \mathcal{P}_3 = -\mathrm{i}\hbar\partial_{x_3} \tag{1.4}$$

$$\boldsymbol{B}(\boldsymbol{x}) = (f(\rho)\cos[2\beta(\theta-\theta_0)], f(\rho)\sin[2\beta(\theta-\theta_0)], g(\rho)) \tag{1.5}$$

$$V(\boldsymbol{x}) = V(\rho), \tag{1.6}$$

where (ρ, θ, x_3) are cylindrical coordinates, the functions $f(\rho)$, $g(\rho)$, $V(\rho)$ are arbitrary and β, θ_0 are free parameters. Therefore, we can decouple the problem as follows: a free motion in the x_3-axis and another motion in the plane (x_1, x_2). In the following, we will restrict ourselves to the analysis of the system in the plane, with Hamiltonian $\mathcal{H}_2 = \mathcal{H}_3 - p_3^2\big/(2m)$.

1.2. Parabolic symmetry

Next, we also assume that there exists a second-order symmetry of parabolic type, that is, with a leading second-order term associated with parabolic coordinates [6],

$$\mathcal{S}_1 = L_3 p_1 + p_1 L_3 + \boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{x}) + N(\boldsymbol{x}), \tag{1.7}$$

where $\boldsymbol{A}(\boldsymbol{x})$ and $N(\boldsymbol{x})$ are the matrix-valued vector and scalar Hermitian functions, respectively. If the magnetic term determined by $f(\rho)$ is present, we arrive, up to an equivalence, at the same model already reported in [1]:

$$\boldsymbol{B}(\boldsymbol{x}) = (x_2/\rho^2, -x_1/\rho^2, 0) \qquad V(\boldsymbol{x}) = 0 \tag{1.8}$$

$$\mathcal{J}_3 = -\mathrm{i}\hbar\partial_\theta + \frac{\hbar}{2}\sigma_3 \equiv L_3 + \frac{\hbar}{2}\sigma_3 \tag{1.9}$$

$$\mathcal{S}_1 = \mathcal{J}_3 p_1 + p_1\mathcal{J}_3 - \mu x_2\boldsymbol{\sigma}\cdot\boldsymbol{B}(\boldsymbol{x}). \tag{1.10}$$

The commutator of $\mathcal{J}_3$ and $\mathcal{S}_1$ gives another second-order symmetry $\mathcal{S}_2$:

$$\mathcal{S}_2 = \mathcal{J}_3 p_2 + p_2\mathcal{J}_3 + \mu x_1\boldsymbol{\sigma}\cdot\boldsymbol{B}(\boldsymbol{x}). \tag{1.11}$$

All these symmetries, together with the Hamiltonian $\mathcal{H}_2$, close the following quadratic algebra:

$$[\mathcal{J}_3, \mathcal{S}_1] = \mathrm{i}\mathcal{S}_2 \qquad [\mathcal{J}_3, \mathcal{S}_2] = -\mathrm{i}\mathcal{S}_1 \qquad [\mathcal{S}_1, \mathcal{S}_2] = -4\mathrm{i}\mathcal{H}_2\mathcal{J}_3. \tag{1.12}$$

As was mentioned in [1], the symmetries $\mathcal{S}_1, \mathcal{S}_2$ are similar to the components of the Laplace–Runge–Lenz vector for the Coulomb potential.

Previous works have considered the quantum mechanical problem that we are dealing with in this paper, mainly in the momentum space [7], but we are also paying attention to some partial aspects in the configuration space [8]. In the present work we want to address it from a self-contained, systematic and more complete point of view, including important properties not considered before. Basically, we will use the SUSY quantum mechanics approach with special emphasis on the shape invariance of the model.

This paper is organized as follows. In section 2, we perform the separation of variables and obtain the discrete symmetries that are basic in the development of the following sections. Next, in section 3 we analyse the factorization and the shape-invariance properties of the radial equation. We study the ground states by means of this factorization in section 4 and the excited states in section 5. In section 6, we study the relationship between the second-order symmetries and the intertwining operators entering the factorization. In section 7, we build the matrix ladder operators connecting eigenstates with different energies of the same Hamiltonian. Finally, in section 8 we conclude this work with some remarks, stressing the most original results obtained in this paper.

2. Separation of variables and discrete symmetries

Let us consider again the two-dimensional matrix Hamiltonian $\mathcal{H}_2$ obtained from (1.1), where we have excluded the free part along the x_3-axis, and where the interaction is given by the magnetic field (1.8), thus, sharing the symmetries (1.9)–(1.12). In order to simplify the expressions, we take

$$\hbar = 2m = 1 \qquad x = \mu x_1 \qquad y = \mu x_2 \qquad r = \mu\rho, \tag{2.1}$$

so that we will work with the following form of the Hamiltonian:

$$\mathcal{H} = \mathcal{H}_2/\mu^2 = -\left(\partial_x^2 + \partial_y^2\right) + \frac{y}{r^2}\sigma_1 - \frac{x}{r^2}\sigma_2, \tag{2.2}$$

where the second-order symmetries are

$$\mathcal{T}_j = \mu^{-1}\mathcal{S}_j \qquad j = 1, 2. \tag{2.3}$$

2.1. Separation of variables

Since the above Hamiltonian commutes with the operator $\mathcal{J}_3$ given in (1.9), we can look for their common eigenfunctions

$$\mathcal{H}\Psi_\lambda = E\Psi_\lambda \qquad \mathcal{J}_3\Psi_\lambda = \lambda\Psi_\lambda \qquad \Psi_\lambda = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{2.4}$$

The rotational symmetry can be used to separate variables in polar coordinates (r, θ) so that the $\mathcal{J}_3$-eigenfunctions in (2.4) are given by

$$\Psi_\lambda(r, \theta) = Y_\lambda(\theta) F_\lambda(r), \tag{2.5}$$

with

$$Y_\lambda(\theta) = \begin{pmatrix}\mathrm{e}^{\mathrm{i}(\lambda-1/2)\theta} & 0\\ 0 & \mathrm{e}^{\mathrm{i}(\lambda+1/2)\theta}\end{pmatrix} \qquad F_\lambda(r) = \begin{pmatrix}f_1(r)\\ f_2(r)\end{pmatrix}. \tag{2.6}$$

Now, taking into account the polar expression for the Laplacian $\Delta = \partial_r^2 + 1/r\partial_r + 1/r^2\partial_\theta^2$, the eigenfunction equation for $\mathcal{H}$ takes the form

$$-F_\lambda''(r) - \frac{1}{r}F_\lambda'(r) + \frac{\lambda^2 + \frac{1}{4}}{r^2}F_\lambda(r) - \frac{\lambda}{r^2}\sigma_3 F_\lambda(r) - \frac{1}{r}\sigma_2 F_\lambda(r) = EF_\lambda(r). \tag{2.7}$$

In order to eliminate the first-order derivative term, we make the replacement

$$F_\lambda(r) = r^{-1/2}\Phi_\lambda(r), \tag{2.8}$$

so that we finally have a one-dimensional matrix Schrödinger-like equation

$$\mathcal{H}_\lambda\Phi_\lambda(r) \equiv \left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\lambda^2 - \lambda\sigma_3}{r^2} - \frac{\sigma_2}{r}\right\}\Phi_\lambda(r) = E\Phi_\lambda(r), \tag{2.9}$$

where we have introduced the notation $\mathcal{H}_\lambda$ for the radial part of the initial Hamiltonian $\mathcal{H}$. Note that this equation is quite similar to the radial Schrödinger equation for a charged particle in a Coulomb potential

$$\mathcal{H}_\ell^c\phi(r) \equiv -\phi''(r) + \frac{\ell^2 - \ell}{r^2}\phi(r) - \frac{1}{r}\phi(r) = E^c\phi(r), \tag{2.10}$$

where ℓ is the orbital momentum. The discrete spectrum is given by the well-known formula

$$E^c = -\frac{1}{4(\ell + n + 1)^2} \qquad n = 0, 1, \ldots. \tag{2.11}$$

2.2. *Discrete symmetries*

In the following we will describe some discrete symmetries of the matrix equation (2.9).

2.2.1. Conjugation. Let us consider the antilinear operator

$$\mathcal{C} = \sigma_3\mathcal{K}, \tag{2.12}$$

where $\mathcal{K}$ is the complex conjugation operator, $\mathcal{K}\Phi(r) = \Phi^*(r)$. Then, it is immediate to check that this is a symmetry of $\mathcal{H}_\lambda$:

$$\mathcal{H}_\lambda\mathcal{C} = \mathcal{C}\mathcal{H}_\lambda. \tag{2.13}$$

The eigenfunctions of $\mathcal{C}$, up to a global phase factor, are of the form

$$\Phi = \begin{pmatrix}\phi_1(r)\\ \mathrm{i}\phi_2(r)\end{pmatrix}, \tag{2.14}$$

where $\phi_1(r)$ and $\phi_2(r)$ are real functions. Hence, from now on, we will choose the eigenfunctions of the matrix equation (2.9) in the form (2.14).

2.2.2. Reflection in λ. It is very easy to see from (2.9) that

$$\sigma_2\mathcal{H}_\lambda = \mathcal{H}_{-\lambda}\sigma_2. \tag{2.15}$$

This means that the eigenfunction problem for $\mathcal{H}_\lambda$ is equivalent to that of $\mathcal{H}_{-\lambda}$:

$$\mathcal{H}_\lambda\Phi_\lambda = E\Phi_\lambda \quad\Longleftrightarrow\quad \mathcal{H}_{-\lambda}\Phi_{-\lambda} = E\Phi_{-\lambda} \qquad \Phi_{-\lambda} = \sigma_2\Phi_\lambda. \tag{2.16}$$

Observe that these two discrete symmetries can also be implemented in the space of eigenfunctions of the two-dimensional equations (2.4). If we compare these discrete symmetries with those of the Coulomb Hamiltonian $\mathcal{H}_\ell^c$ of (2.10), we see that the conjugation is translated into the real character of the eigenfunctions, while the reflection property means that $\mathcal{H}_\ell^c = \mathcal{H}_{-\ell+1}^c$.

3. Factorizations and shape invariance

In this section, we will investigate the factorization and the supersymmetrical properties of (2.9), the radial part of the Pauli equation. This one-dimensional 2×2 matrix problem is interesting by itself, for an arbitrary value of λ, but especially for the two-dimensional physical model (2.2) with $\lambda = 1/2 + m, m \in \mathbb{Z}$. In particular, we will see that this matrix model obeys the shape-invariance properties. This is very interesting because, up to now, the shape invariance was used as a very elegant algebraic approach in the solution of one- and two-dimensional scalar spectral problems [2, 3, 9, 10], while in this section it will be used to determine the spectrum of a one-dimensional matrix example.

Following closely the well-known factorization of the Coulomb Hamiltonian (2.10), we propose here the following ansatz for this matrix case:

$$\mathcal{H}_\lambda = \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{A + B\sigma_3}{r} + D\sigma_2\right)\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{A + B\sigma_3}{r} + D\sigma_2\right) + \gamma, \tag{3.1}$$

where A, B, D and γ are real constants to be determined. Indeed, we find two different solutions having the following form:

$$\mathcal{H}_\lambda = L_\lambda^- L_\lambda^+ + \gamma_\lambda = L_{\lambda-1}^+ L_{\lambda-1}^- + \gamma_{\lambda-1} \qquad \gamma_\lambda = \frac{-1}{4(\lambda + 1/2)^2}, \tag{3.2}$$

where we have used the following notation for the operators:

$$L_\lambda^\pm = \mp\frac{\mathrm{d}}{\mathrm{d}r} + \frac{(\lambda + 1/2) - (1/2)\sigma_3}{r} - \frac{(1/2)\sigma_2}{(\lambda + 1/2)} \equiv \mp\frac{\mathrm{d}}{\mathrm{d}r} + W_\lambda(r), \tag{3.3}$$

$W_\lambda(r)$ being the matrix superpotentials.

Of course, equations (3.1)–(3.3) are valid for any value of $\lambda \neq -1/2$, and therefore we can build a hierarchy of Hamiltonians $\{\mathcal{H}_{\lambda+m}\}$, $m \in \mathbb{Z}$, as follows:

$$\mathcal{H}_{\lambda+m} = L_{\lambda+m}^- L_{\lambda+m}^+ + \gamma_{\lambda+m} = L_{\lambda+m-1}^+ L_{\lambda+m-1}^- + \gamma_{\lambda+m-1}. \tag{3.4}$$

These relations allow us to define an algebra of operators of the Hamiltonian hierarchy [11]. The operators $L_{\lambda+m}^\pm$ act as intertwining operators between two consecutive Hamiltonians $\mathcal{H}_{\lambda+m}$ and $\mathcal{H}_{\lambda+m+1}$:

$$L_{\lambda+m}^+ \mathcal{H}_{\lambda+m} = \mathcal{H}_{\lambda+m+1} L_{\lambda+m}^+ \qquad \mathcal{H}_{\lambda+m} L_{\lambda+m}^- = L_{\lambda+m}^- \mathcal{H}_{\lambda+m+1}. \tag{3.5}$$

These relations imply that the eigenfunctions of $\mathcal{H}_{\lambda+m}$ can be obtained from those of $\mathcal{H}_{\lambda+m+1}$ with the same eigenvalue by acting with $L_{\lambda+m}^-$, while the operator $L_{\lambda+m}^+$ makes this connection in the opposite way. This correspondence between square-integrable eigenfunctions is one to one, except when $L_{\lambda+m}^\pm$ annihilates some eigenfunctions. As usual, relations (3.4) and (3.5) mean that the Hamiltonians of the hierarchy $\{\mathcal{H}_{\lambda+m}\}$ are shape invariant: intertwined Hamiltonians have the same shape but different values of the parameter.

Now, let us see the effect of the discrete symmetries $\mathcal{C}$ and σ_2 on the intertwining operators. It is easy to check that

$$\mathcal{C} L_{\lambda+m}^\pm = L_{\lambda+m}^\pm \mathcal{C} \tag{3.6}$$

and

$$\sigma_2 L_{\lambda+m}^+ = L_{-\lambda-m-1}^- \sigma_2. \tag{3.7}$$

From the commutation (3.6), we see that the intertwining operators keep the form of the eigenfunctions Φ in (2.14). Relation (3.7) will have consequences for the ground states, as we will see in the next section.

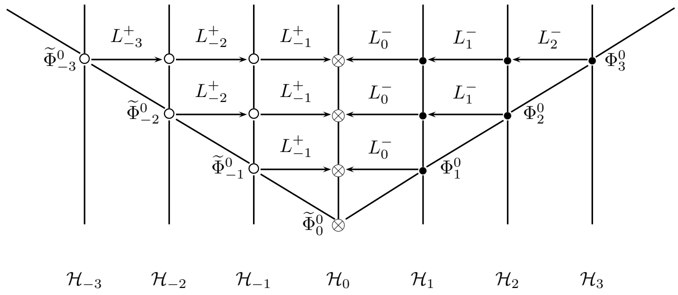

Figure 1. Energy level diagram for the 'integer hierarchy', including the information about the ground states and the operators $L^{\pm}_k$.

The hierarchies obtained from values λ and $\lambda + m_0$, where $m_0 \in \mathbb{Z}$, are the same, so we can get a family of different hierarchies characterized, for instance, by the values $\lambda \in [0, 1/2]$. Although the initial two-dimensional model (2.2) is related to the hierarchy with half-integer values of λ, we will analyse the properties for any λ. Thus, we can distinguish the following three classes of hierarchies.

- $\lambda = 0$. We call it the '*integer hierarchy*' $\{\mathcal{H}_m\}$, $m \in \mathbb{Z}$. It includes the Hamiltonian $\mathcal{H}_0$ and would correspond to two-valued eigenfunctions in the context of the two-dimensional problem (2.2). The reflection in λ is implemented in the hierarchy, a scheme of which can be seen in figure 1.
- $\lambda = 1/2$. We refer to this case as the '*physical hierarchy*' $\{\mathcal{H}_{m+1/2}\}$, $m \in \mathbb{Z}$, because it is associated with the physical spin-1/2 systems of equation (2.2). Indeed, it is the only case considered in all the previous references [1, 7, 8], and for it the operators $L^{\pm}_{-1/2}$ are not well defined, as can be seen from (3.3). Therefore, there are no first-order intertwining operators connecting the Hamiltonians $\mathcal{H}_{1/2}$ and $\mathcal{H}_{-1/2}$. Remark that the hierarchy also includes the reflection in λ.
- $0 < \lambda < 1/2$. We will refer to this as the '*general hierarchy*' $\{\mathcal{H}_{\lambda+m}\}$. It does not implement the reflection in λ; in fact this reflection gives rise to the hierarchies with values $-1/2 < \lambda < 0$. The associated two-dimensional eigenfunctions are multiple-valued.

4. The ground states of the hierarchies

It is known [2, 9] that the ground states of SUSY-hierarchy Hamiltonians are the main elements when using the shape-invariance approach in the construction of the whole spectra and the eigenfunctions for one-dimensional scalar problems. In this section, the explicit expressions for the ground states of the matrix SUSY hierarchies described in the previous section will be found and analysed.

Let us start with the ground-state wavefunctions $\Phi^0_{\lambda+m}$ annihilated by the general intertwining operator $L^+_{\lambda+m}$:

$$L^+_{\lambda+m}\Phi^0_{\lambda+m} = 0. \tag{4.1}$$

Then, according to (3.2) and (3.4), this will be an eigenfunction of $\mathcal{H}_{\lambda+m}$ with energy $E^0_{\lambda+m} = \gamma_{\lambda+m}$:

$$\mathcal{H}_{\lambda+m}\Phi^0_{\lambda+m} = E^0_{\lambda+m}\Phi^0_{\lambda+m} \qquad E^0_{\lambda+m} = -\frac{1}{4(\lambda+m+1/2)^2}. \tag{4.2}$$

Let us use the notation (2.14) for the ground state

$$\Phi^0_{\lambda+m} = \begin{pmatrix} \phi^0_{1,\lambda+m}(r) \\ \mathrm{i}\phi^0_{2,\lambda+m}(r) \end{pmatrix}, \tag{4.3}$$

and let us make the substitution

$$\phi^0_{j,\lambda}(r) = z^{1+\lambda+m}\varphi_j(z) \quad j = 1, 2 \qquad z = \frac{r}{2(\lambda + m + 1/2)} \tag{4.4}$$

in equation (4.1). Then, we arrive at a modified Bessel equation for the component $\varphi_1(z)$,

$$z^2\varphi_1''(z) + z\varphi_1'(z) - (z^2 + 1)\varphi_1(z) = 0, \tag{4.5}$$

with general solution

$$\varphi_1(z) = c_1 I_1(z) + c_2 K_1(z), \tag{4.6}$$

where $I_1(z)$, $K_1(z)$ are the two linearly independent modified Bessel functions. After a simple calculation, we arrive at the general form of the ground state (4.1) for any integer m:

$$\Phi^0_{\lambda+m}(r) = c_1\mathtt{K}_{\lambda+m}(r) + c_2\mathtt{I}_{\lambda+m}(r), \tag{4.7}$$

with

$$\mathtt{K}_{\lambda+m}(r) = r^{1+\lambda+m}\begin{pmatrix} K_1\left(\frac{r}{2(\lambda+m+1/2)}\right) \\ \mathrm{i}K_0\left(\frac{r}{2(\lambda+m+1/2)}\right) \end{pmatrix} \tag{4.8}$$

$$\mathtt{I}_{\lambda+m}(r) = r^{1+\lambda+m}\begin{pmatrix} I_1\left(\frac{r}{2(\lambda+m+1/2)}\right) \\ -\mathrm{i}I_0\left(\frac{r}{2(\lambda+m+1/2)}\right) \end{pmatrix}. \tag{4.9}$$

Remark that as the asymptotic behaviour of the modified Bessel functions $I_0(r)$ and $I_1(r)$ in the limit $r \to +\infty$ exponentially increases, they cannot lead to square-integrable functions. On the other hand, in the same limit $K_0(r)$ and $K_1(r)$ decrease exponentially, while near the origin $K_0(r) \approx \log(r)$ and $K_1(r) \approx 1/r$. Thus, the physical solution will be $\Phi^0_{\lambda+m}(r) \propto \mathtt{K}_{\lambda+m}(r)$, which is a well-behaved ground state of $\mathcal{H}_{\lambda+m}$ with $\lambda + m > 0$ (the ground state of $\mathcal{H}_0$ will be considered later).

Let us now consider the ground states $\widetilde{\Phi}^0_{\lambda+m}$ of $\mathcal{H}_{\lambda+m}$ with $\lambda + m < 0$. They are annihilated by the operator $L^-_{\lambda+m-1}$ in the factorization (3.2): $L^-_{\lambda+m-1}\widetilde{\Phi}^0_{\lambda+m} = 0$. Then, according to (3.2) and (3.4), this will be an eigenfunction with energy $\widetilde{E}^0_{\lambda+m} = E^0_{-\lambda-m} = \gamma_{-\lambda-m}$:

$$\mathcal{H}_{\lambda+m}\widetilde{\Phi}^0_{\lambda+m} = \widetilde{E}^0_{\lambda+m}\widetilde{\Phi}^0_{\lambda+m} \qquad \widetilde{E}^0_{\lambda+m} = -\frac{1}{4(\lambda + m - 1/2)^2}. \tag{4.10}$$

After a computation similar to the one carried out in the previous case, we get the following general solutions:

$$\widetilde{\Phi}^0_{\lambda+m}(r) = c_1 r^{1-\lambda-m}\begin{pmatrix} K_0\left(\frac{-r}{2(\lambda+m-1/2)}\right) \\ \mathrm{i}K_1\left(\frac{-r}{2(\lambda+m-1/2)}\right) \end{pmatrix} + c_2 r^{1-\lambda-m}\begin{pmatrix} I_0\left(\frac{-r}{2(\lambda+m-1/2)}\right) \\ -\mathrm{i}I_1\left(\frac{-r}{2(\lambda+m-1/2)}\right) \end{pmatrix}. \tag{4.11}$$

As in the previous case, a study of the asymptotic behaviour shows that the square-integrable and finite ground states are obtained by taking $c_2 = 0$ in (4.11), whenever $\lambda + m < 0$. The results for this case could be found directly from the previous one with the help of the relation (3.7) between both types of intertwining operators through σ_2.

In summary, we have one reasonable ground state for each Hamiltonian $\mathcal{H}_{\lambda+m}$: if $\lambda + m > 0$ it is of the form $\Phi^0_{\lambda+m}(r)$, while if $\lambda + m < 0$ it will take the form $\widetilde{\Phi}^0_{\lambda+m}(r)$.

There is only one Hamiltonian, $\mathcal{H}_0$, having a doubly degenerated square-integrable solution with energy $E_0^0 = -1$. But this case is rather special and we will comment on its lack of physical meaning later on.

To end this section, let us now discuss the relationship between these ground-state solutions and the superpotential. First of all, let us recall that

$$L^+_{\lambda+m}\mathtt{K}_{\lambda+m}(r) = L^+_{\lambda+m}\mathtt{I}_{\lambda+m}(r) = 0. \tag{4.12}$$

Let us now introduce the 2×2 matrix $\mathbb{M}_{\lambda+m}(r)$, whose columns are $\mathtt{K}_{\lambda+m}(r)$ and $\mathtt{I}_{\lambda+m}(r)$, in this order. Then, equations (4.12) can be expressed in a more compact form:

$$L^+_{\lambda+m}\mathbb{M}_{\lambda+m}(r) = -\mathbb{M}'_{\lambda+m}(r) + W_{\lambda+m}(r)\mathbb{M}_{\lambda+m}(r) = 0. \tag{4.13}$$

Thus, after substituting (4.7), we again get the expression of (3.3) for the matrix superpotential in terms of the solution matrix:

$$W_{\lambda+m}(r) = \mathbb{M}'_{\lambda+m}(r)\mathbb{M}^{-1}_{\lambda+m}(r) = \frac{\left(\lambda + m + \frac{1}{2}\right) - \frac{1}{2}\sigma_3}{r} - \frac{\frac{1}{2}\sigma_2}{\lambda + m + \frac{1}{2}}. \tag{4.14}$$

Therefore, we have shown that this class of matrix solutions $\mathbb{M}_{\lambda+m}(r)$ gives rise in a non-trivial way to a Hermitian matrix superpotential. It is important to stress that there are not many explicit examples satisfying this condition [12].

5. Excited states

In this section we will study the excited states of the system (2.9). There are some general expressions valid for any hierarchy. In order to find the n-excited state $\Phi^n_\lambda(r)$ of a particular Hamiltonian $\mathcal{H}_\lambda$, $\lambda > 0$, we can start with the ground state $\Phi^0_{\lambda+n}(r)$, $n \in \mathbb{Z}^+$, of the Hamiltonian $\mathcal{H}_{\lambda+n}$ in the same hierarchy. Then

$$\Phi^n_\lambda(r) = L^-_\lambda L^-_{\lambda+1} \cdots L^-_{\lambda+n-1}\Phi^0_{\lambda+n}(r) \tag{5.1}$$

with energy

$$E^n_\lambda = -\frac{1}{4(\lambda + n + 1/2)^2}. \tag{5.2}$$

Two natural questions now arise:

- Are the excited states obtained in this way always square integrable?
- Are these excited states the only bounded physical states for each Hamiltonian of the hierarchy?

Here, in the context of the factorization method, we can answer to the first question: under the assumption $\lambda > 0$, all the eigenfunctions build in the form (5.1) are finite, square integrable and vanishing at the endpoints (0 and $+\infty$), so they indeed describe excited bound states. The details are given in the appendix. With respect to the second question, as in the two-dimensional scalar models [10], it should be studied from the point of view of a general 'oscillation theorem' for matrix Sturm–Liouville operators to guarantee that no additional excited states exist besides those obtained after applying the shape-invariance procedure (for a discussion on two-dimensional scalar models see [10]). Meanwhile, one can note that each component of the constructed nth excited states (5.1) has exactly $(n - 1)$ nodes.

If $\lambda < 0$, the excited states with similar properties are obtained from the other class of ground states:

$$\widetilde{\Phi}^n_\lambda(r) = L^+_{\lambda-1}L^+_{\lambda-2} \cdots L^+_{\lambda-n}\widetilde{\Phi}^0_{\lambda-n}(r), \tag{5.3}$$

with energy

$$\widetilde{E}_\lambda^n = -\frac{1}{4(\lambda - n - 1/2)^2}. \tag{5.4}$$

Now, we will consider some specific features of each hierarchy.

5.1. 'Integer hierarchy' ($\lambda = 0$): $\mathcal{H}_m, m \in \mathbb{Z}$

A representation of this hierarchy can be seen in figure 1. It includes the Hamiltonian $\mathcal{H}_0$ which has the peculiarity that both procedures lead to two square-integrable excited states for each energy

$$E_0^n = -\frac{1}{4(n + 1/2)^2}. \tag{5.5}$$

Therefore, in principle, each eigenvalue level is doubly degenerated. However, an important difference is that in each eigenfunction one of the components vanishes at the origin, while the other one does not. This feature leads to some problems about the physical interpretation, which we will discuss now.

The matrix Hamiltonian $\mathcal{H}_0$ in (2.9) can be diagonalized by means of the unitary matrix $\mathcal{U} = \frac{1}{\sqrt{2}}(\mathrm{i}\sigma_0 + \sigma_1)$,

$$\mathcal{H}_0\Phi = \left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\sigma_2}{r}\right\}\Phi \quad\Longleftrightarrow\quad \mathcal{H}_d\Phi_d = \left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\sigma_3}{r}\right\}\Phi_d, \tag{5.6}$$

where $\mathcal{H}_d = \mathcal{U}\mathcal{H}_0\mathcal{U}^\dagger$ and $\Phi_d = \mathcal{U}\Phi$. Thus, we can write

$$\mathcal{H}_d = \begin{pmatrix} h_- & 0 \\ 0 & h_+ \end{pmatrix}, \tag{5.7}$$

where $h_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} \pm \frac{1}{r}$ are scalar Hamiltonians. In particular, the lowest energy ($E = -1$) normalizable solutions of the Schrödinger equation with Hamiltonian (5.7) are as follows:

$$\Phi_-^0 = \frac{r}{\sqrt{2}}\begin{pmatrix} K_1(r) + K_0(r) \\ 0 \end{pmatrix} \qquad \Phi_+^0 = \frac{r}{\sqrt{2}}\begin{pmatrix} 0 \\ K_1(r) - K_0(r) \end{pmatrix} \tag{5.8}$$

$$\Phi_d^0 = \mathcal{U}\Phi_0^0 \qquad \widetilde{\Phi}_d^0 = \mathcal{U}\widetilde{\Phi}_0^0. \tag{5.9}$$

One can note that these solutions, as well as the higher energy solutions, do not vanish at the origin and therefore correspond to divergent mean values of the potential energy terms. A plot of some of these functions can be seen in figure 2. For this reason, solutions (5.9), though being normalizable, *do not* belong to the physical sector of the model.

Nevertheless, these non-physical solutions are quite useful from the point of view of SUSY quantum mechanics. Indeed, they are used to generate the intertwining operators $L_m^\pm$ to get a hierarchy $\mathcal{H}_m$ of shape-invariant non-diagonal Hamiltonians starting from $\mathcal{H}_0$, which is diagonal. Besides, any $\mathcal{H}_m, m \neq 0$, has a sensible physical discrete spectrum with eigenfunctions vanishing at the origin. By extending the label m to a real number we get all the hierarchies $\mathcal{H}_{\lambda+m}$.

5.2. 'Physical hierarchy' ($\lambda = 1/2$): $\mathcal{H}_{m+1/2}, m \in \mathbb{Z}$

The general scheme (5.1)–(5.4) is still valid here. But since in this hierarchy the operators $L_{-1/2}^\pm$ are not defined, in order to connect the excited states of $\mathcal{H}_{1/2}$ and $\mathcal{H}_{-1/2}$ we need a zero-order intertwining operator given by the reflection (2.15),

$$\Phi_{1/2}^n \propto \sigma_2\Phi_{-1/2}^n. \tag{5.10}$$

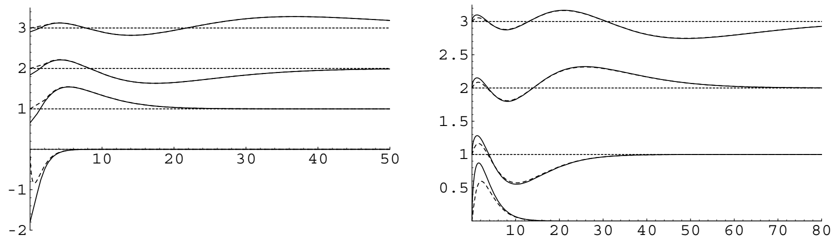

Figure 2. Plot of the two components of the eigenfunctions (the first component in the solid line, the second one in the dashed line) for the ground and the first excited states of the Hamiltonians $\mathcal{H}_0$ (left) and $\mathcal{H}_{1/2}$ (right).

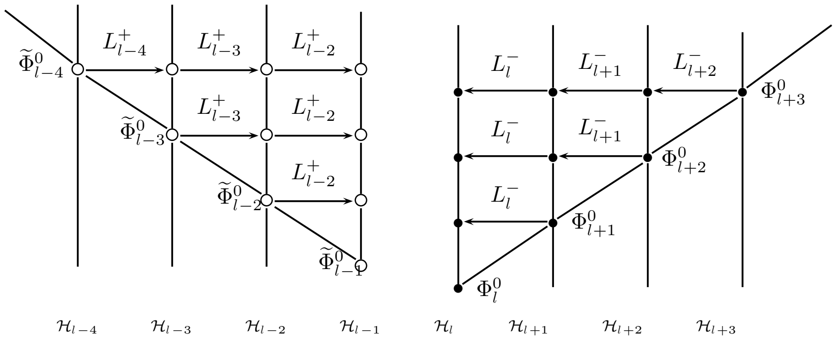

Figure 3. Energy level diagram for the 'general hierarchy', including the information about the ground states and the operators $L^{\pm}_{\lambda+m}$.

It is interesting to remark that it is possible to include in a very natural way the operator σ_2 inside the set of intertwining operators $L^{\pm}_{\lambda+m}$ if we 'normalize' them by a factor

$$\widetilde{L}^{\pm}_{\lambda+m} := (\lambda + m + 1/2)L^{\pm}_{\lambda+m}. \tag{5.11}$$

Then, the set $\left\{\widetilde{L}^{\pm}_{\lambda+m}\right\}$, $m \in \mathbb{Z}$, will also act as intertwining operators of the hierarchy $\mathcal{H}_{\lambda+m}$ as in (3.5), but they are always well defined, and in particular $\widetilde{L}^{\pm}_{-1/2} = -\sigma_2$, as it should be.

However, the expression of the Hamiltonians in terms of the normalized operators are changed by these factors and it gives rise to the following 'commutation rules':

$$\widetilde{L}^{+}_{\lambda+m-1}\widetilde{L}^{-}_{\lambda+m-1} - \widetilde{L}^{-}_{\lambda+m}\widetilde{L}^{+}_{\lambda+m} = -2(\lambda + m)\mathcal{H}_{\lambda+m}. \tag{5.12}$$

5.3. *'General hierarchy'* $\mathcal{H}_{\lambda+m}$, $m \in \mathbb{Z}$

If $0 < \lambda < 1/2$ we will have two kinds of Hamiltonians in the hierarchy. The right-hand Hamiltonians $\mathcal{H}_{\lambda+m}$, $m = 0, 1, \ldots$, and the left-hand ones $\mathcal{H}_{\lambda-m}$, $m \in \mathbb{N}$. We have two special features: (a) the excited states of the right (left) Hamiltonians are only obtained from the right (left) ground states. However $L^{\pm}_{\lambda-1}$, do not connect these two kinds of Hamiltonians, nor the reflection matrix σ_2 can be used to connect the two Hamiltonian sectors. (b) The spectrum of these two types of Hamiltonians is different. In this case the operators $L^{\pm}_{\lambda+m}$, acting on physical states, do not always generate again physical states. A schematic diagram of this case can be seen in figure 3.

6. Factorization operators and second-order symmetries

As we have seen above, the action of the operators $L^{\pm}_{\lambda+m}$ on eigenfunctions of the Hamiltonian hierarchy changes the label $\lambda + m$ in one unit, while keeping the energy. We can write this property as follows:

$$L^{+}_{\lambda+m}\Phi_{\lambda+m}(r) \propto \Phi_{\lambda+m+1}(r) \qquad L^{-}_{\lambda+m}\Phi_{\lambda+m+1}(r) \propto \Phi_{\lambda+m}(r). \tag{6.1}$$

If we recall that the label $\lambda+m$ is for the $\mathcal{J}_3$-eigenvalue, we can say that the operators $L^{\pm}_{\lambda+m}$ act essentially as lowering or raising operators for $\mathcal{J}_3$. However, when they act on the eigenstates of the Hamiltonians $\mathcal{H}_\lambda$ they preserve the energy eigenvalue.

Now, if we go back to the second-order symmetries $\mathcal{T}_1, \mathcal{T}_2$ of the initial Hamiltonian $\mathcal{H}$, and we introduce the operators $\mathcal{T}^{\pm} = \pm\mathrm{i}(\mathcal{T}_1 \mp \mathrm{i}\mathcal{T}_2)/2$, they satisfy the commutation rules

$$[\mathcal{J}_3, \mathcal{T}^{\pm}] = \pm\mathcal{T}^{\pm} \qquad [\mathcal{H}, \mathcal{T}^{\pm}] = 0 \qquad [\mathcal{T}^{+}, \mathcal{T}^{-}] = -2\mathcal{H}\mathcal{J}_3. \tag{6.2}$$

This means that $\mathcal{T}^{\pm}$ acting on the common eigenfunctions of $\mathcal{H}$ and $\mathcal{J}_3$ realizes the same role played by $L^{\pm}_{\lambda+m}$: change the $\mathcal{J}_3$-eigenvalue in one unit while they leave that of $\mathcal{H}$ unaltered. Therefore, there must be a close relationship between both types of operators, as it was the case for the Coulomb problem [13]. Indeed, if we write the eigenfunctions $\Psi_\lambda(r,\theta)$ given in (2.5)–(2.6) in terms of $\Phi_\lambda(r)$ introduced in (2.8),

$$\Psi_\lambda(r,\theta) = r^{-1/2}Y_\lambda(\theta)\Phi_\lambda(r), \tag{6.3}$$

and we use the first commutation relation in equation (6.2), we have

$$\mathcal{T}^{\pm}\Psi_\lambda(r,\theta) \propto \Psi_{\lambda\pm1}(r,\theta). \tag{6.4}$$

In order to prove this relationship explicitly, let us first use (1.10)–(1.11) and (2.1)–(2.3) to express $\mathcal{T}^{\pm}$ in polar coordinates:

$$\begin{aligned}\mathcal{T}^{+} &= -\frac{\mathrm{i}}{2}\,\mathrm{e}^{\mathrm{i}\theta}\left\{-2\left(\frac{\mathrm{i}}{r}\partial_\theta + \partial_r\right)\left(\partial_\theta + \frac{\mathrm{i}}{2}(\sigma_3+\sigma_0)\right) + \begin{pmatrix}0 & -\mathrm{e}^{-\mathrm{i}\theta}\\ \mathrm{e}^{\mathrm{i}\theta} & 0\end{pmatrix}\right\}\\ \mathcal{T}^{-} &= \frac{\mathrm{i}}{2}\left\{-2\left(-\frac{\mathrm{i}}{r}\partial_\theta + \partial_r\right)\left(\partial_\theta + \frac{\mathrm{i}}{2}(\sigma_3+\sigma_0)\right) - \begin{pmatrix}0 & -\mathrm{e}^{-\mathrm{i}\theta}\\ \mathrm{e}^{\mathrm{i}\theta} & 0\end{pmatrix}\right\}\mathrm{e}^{-\mathrm{i}\theta}.\end{aligned} \tag{6.5}$$

Then, if we take into account that

$$\partial_\theta\Psi_\lambda(r,\theta) = \mathrm{i}r^{-1/2}Y_\lambda(\theta)\left(\lambda\sigma_0 - \frac{1}{2}\sigma_3\right)\Phi_\lambda(r) \tag{6.6}$$

$$\partial_r\Psi_\lambda(r,\theta) = r^{-1/2}Y_\lambda(\theta)\left(\frac{-1}{2r} + \partial_r\right)\Phi_\lambda(r) \tag{6.7}$$

$$\begin{pmatrix}0 & -\mathrm{e}^{-\mathrm{i}\theta}\\ \mathrm{e}^{\mathrm{i}\theta} & 0\end{pmatrix}\Psi_\lambda(r,\theta) = (-\mathrm{i})r^{-1/2}Y_\lambda(\theta)\sigma_2\Phi_\lambda(r) \tag{6.8}$$

and we insert these identities in (6.5), we get the induced action of $\mathcal{T}^{\pm}$ on $\Phi_\lambda(r)$:

$$\mathcal{T}^{\pm}\Phi_\lambda(r) = \left(\lambda + \frac{1}{2}\right)\left[\mp\frac{\mathrm{d}}{\mathrm{d}r} + \frac{(\lambda\pm1/2) - (1/2)\sigma_3}{r} - \frac{1/2}{\lambda\pm1/2}\sigma_2\right]\Phi_\lambda(r). \tag{6.9}$$

If we compare the above expression with definition (3.3) of $L^{\pm}_{\lambda}$, we see that, indeed, the action of the second-order symmetries $\mathcal{T}^{\pm}$ induced in the space of the eigenfunctions $\Phi_{\lambda+m}(r)$ essentially coincides with the action of the intertwining operators $L^{\pm}_{\lambda+m}$, as we expected from the initial intuitive arguments. In fact, we get the normalized operators $\widetilde{L}^{\pm}_{\lambda}$ defined in (5.11), and commutation (6.2) of $\mathcal{T}^{\pm}$ induces in the space of functions $\Phi_{\lambda+m}(r)$ the relation displayed in (5.12) between the operators $\widetilde{L}^{\pm}_{\lambda+m}$.

7. Ladder operators

In this section we will investigate the construction of ladder operators that will allow us to link the excited states of a given Hamiltonian $\mathcal{H}_\lambda$.

Let us consider the eigenvalue equation (2.9) for a given Hamiltonian $\mathcal{H}_\lambda$ corresponding to the energy E_λ^n and eigenfunction $\Phi_\lambda^n(r)$ given in (5.1)–(5.2). After multiplying by r^2, we reorder the terms in the following way:

$$\mathcal{R}_{\lambda,n}\Phi_\lambda^n(r) \equiv \left[-r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} - r\sigma_2 + \frac{r^2}{4(\lambda+n+1/2)} - \lambda\sigma_3\right]\Phi_\lambda^n(r) = -\lambda^2\Phi_\lambda^n(r). \tag{7.1}$$

In a similar way to what we did in (3.1)–(3.2), we can factorize the operators $\mathcal{R}_{\lambda,n}$ in two ways:

$$\mathcal{R}_{\lambda,n} = Q^+_{\lambda,n}Q^-_{\lambda,n} + \omega_{\lambda,n} = Q^-_{\lambda,n+1}Q^+_{\lambda,n+1} + \omega_{\lambda,n+1}, \tag{7.2}$$

where

$$Q^+_{\lambda,n} = \left[-r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda\sigma_3}{2(\lambda+n)} - (\lambda+n) + \frac{1}{2} + \frac{\sigma_2}{2(\lambda+n+1/2)}r\right]\mathcal{D}^{-1}_{\lambda+n} \tag{7.3}$$

$$Q^-_{\lambda,n} = \mathcal{D}_{\lambda+n}\left[r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda\sigma_3}{2(\lambda+n)} - (\lambda+n) - \frac{1}{2} + \frac{\sigma_2}{2(\lambda+n+1/2)}r\right] \tag{7.4}$$

$$\omega_{\lambda,n} = -\frac{\lambda^2}{4(\lambda+n)^2} - (\lambda+n)^2 + \frac{1}{4}, \tag{7.5}$$

and the operators $\mathcal{D}_{\lambda+n}$ are dilation operators defined by

$$\mathcal{D}_{\lambda+n}r = \frac{\lambda+n+1/2}{\lambda+n-1/2}r = \sqrt{\frac{E_\lambda^n}{E_\lambda^{n-1}}}r. \tag{7.6}$$

The operators $Q^\pm_{\lambda,n+1}$ act as intertwining of the differential operators $\mathcal{R}_{\lambda,n}$ and $\mathcal{R}_{\lambda,n+1}$:

$$Q^+_{\lambda,n+1}\mathcal{R}_{\lambda,n} = \mathcal{R}_{\lambda,n+1}Q^+_{\lambda,n+1} \qquad Q^-_{\lambda,n+1}\mathcal{R}_{\lambda,n+1} = \mathcal{R}_{\lambda,n}Q^-_{\lambda,n+1}. \tag{7.7}$$

As a consequence, the set of operators $\{Q^\pm_{\lambda,n}, n \in \mathbb{Z}^+\}$ will act as lowering and raising operators for the Hamiltonian $\mathcal{H}_\lambda$,

$$Q^+_{\lambda,n+1} : \Phi_\lambda^n \to \Phi_\lambda^{n+1} \qquad Q^-_{\lambda,n+1} : \Phi_\lambda^{n+1} \to \Phi_\lambda^n. \tag{7.8}$$

From (7.2) we can compute the normalization of the eigenfunctions obtained in this way:

$$\left\|\Phi_\lambda^{n+1}\right\|^2 = (\omega_{\lambda,0} - \omega_{\lambda,n+1})\left\|\Phi_\lambda^n\right\|^2. \tag{7.9}$$

In particular for $n = 0$, we have $Q_0^-\Phi_\lambda^0 = 0$. The operators $Q^\pm_{\lambda,n}$ constitute a non-trivial generalization of those well known for the Coulomb problem [11].

Finally, we can join the two families of 'ladder operators' found up to now, $\{L^\pm_{\lambda+m}\}$ and $\{Q^\pm_n\}$, to build a 'dynamical algebra' inside the hierarchy $\{\mathcal{H}_{\lambda+m}\}$. However, only for the case $\lambda = 1/2$ all these operators will connect exclusively physical states of the discrete spectrum.

8. Remarks and conclusions

In this paper we have studied a two-dimensional Pauli Hamiltonian, which has two independent integrals of motion, from the point of view of supersymmetric quantum mechanics. We have examined a series of properties that were not fully explored up to now:

(i) We have considered some useful discrete symmetries since the very beginning.

(ii) We also included in our study a family of Hamiltonian hierarchies labelled by the parameter $\lambda \in [0, 1/2]$. For all these hierarchies the spectral problem was solved by means of the matrix shape-invariant method.

(iii) We have shown the relation between these intertwining operators and the second-order symmetries.

(iv) We also computed the ladder operators suitable to this matrix problem, which link the excited states of the same Hamiltonian.

In summary, we have shown that the methods employed along with this paper (SUSY quantum mechanics) constitute a very useful tool for the investigation of Pauli matrix Hamiltonians and deserve to be exploited for more general situations.

Acknowledgments

This work has been partially supported by Spanish Ministerio de Educación y Ciencia under projects BMF2002-02000, BMF2002-03773, SAB2004-0143 (sabbatical grant of MVI), Ministerio de Asuntos Exteriores (AECI grant 0000147625 of SK) and Junta de Castilla y León (Excellence Project VA013C05). The research of MVI is also supported by the Russian grants RFFI 06-01-00186-a, RSS 5538.2006.2 and RNP 2.1.1.1112. The authors would like to thank Professor G P Pron'ko for bringing this problem to their attention.

Appendix

In this appendix we will show that the excited physical states $\Phi^n_\lambda(r)$ are well behaved near the origin and at infinity. We start with expression (5.1) for the excited states, taking into account that the physical ground states are $\Phi^0_{\lambda+n}(r) \propto \mathtt{K}_{\lambda+n}(r)$, where $\mathtt{K}_{\lambda+n}(r)$ is given in (4.8). For the sake of simplicity, we will use in the following the variable $a = \lambda + n$, and therefore (5.1) takes the form

$$\Phi^n_{a-n}(r) = L^-_{a-n} L^-_{a-n+1} \cdots L^-_{a-2} L^-_{a-1} \Phi^0_a(r), \tag{A.1}$$

where, according to (4.8), the ground state is given by

$$\Phi^0_a(r) = (2a+1)^{a+1} \begin{pmatrix} z^{a+1} K_1(z) \\ \mathrm{i} z^{a+1} K_0(z) \end{pmatrix} \qquad z = \frac{r}{2a+1}. \tag{A.2}$$

Using the new variable z, the operators L^-_b adopt the form

$$L^-_b = \frac{1}{2a+1} \left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{2b+1-\sigma_3}{2z} - \frac{2a+1}{2b+1} \sigma_2 \right]. \tag{A.3}$$

It is interesting to know the approximate form that have these operators near the origin $(r, z \approx 0)$ and at infinity $(r, z \approx \infty)$. Indeed, we have the following:

$$\text{if} \quad r, z \approx 0, \quad L^-_b \approx N^-_b = \frac{1}{2a+1} \begin{pmatrix} \partial_z + \frac{b}{z} & 0 \\ 0 & \partial_z + \frac{b+1}{z} \end{pmatrix} \tag{A.4}$$

$$\text{if} \quad r, z \approx \infty, \quad L^-_b \approx G^-_b = \frac{1}{2a+1} \begin{pmatrix} \partial_z & \mathrm{i}\frac{2a+1}{2b+1} \\ -\mathrm{i}\frac{2a+1}{2b+1} & \partial_z \end{pmatrix}. \tag{A.5}$$

A.1. Behaviour near the origin

In this approximation, we have that

$$\Phi^n_{a-n} \approx N^-_{a-n} N^-_{a-n+1} \cdots N^-_{a-2} N^-_{a-1} \Phi^0_a \qquad n = 1, 2, \ldots \tag{A.6}$$

and using

$$K_0(z) \approx \log(z) \qquad K_1(z) \approx 1/z \tag{A.7}$$

$$K_0'(z) = -K_1(z) \qquad zK_1'(z) = -zK_0(z) - K_1(z), \tag{A.8}$$

we can prove that

$$N^-_{a-1}\Phi^0_a \approx (2a+1)^a \begin{pmatrix} 2a-1 & 0 \\ 0 & 2a+1 \end{pmatrix} \begin{pmatrix} z^a K_1(z) \\ \mathrm{i} z^a K_0(z) \end{pmatrix}$$

$$N^-_{a-2}N^-_{a-1}\Phi^0_a \approx (2a+1)^{a-1} \begin{pmatrix} [2a-1][2a-3] & 0 \\ 0 & [2a+1][2a-1] \end{pmatrix} \begin{pmatrix} z^{a-1} K_1(z) \\ \mathrm{i} z^{a-1} K_0(z) \end{pmatrix}.$$

Therefore

$$\Phi^n_{a-n} \approx 2^n (2a+1)^{a-n+1} \begin{pmatrix} \frac{\Gamma(a+1/2)}{\Gamma(a+1/2-n)} & 0 \\ 0 & \frac{\Gamma(a+3/2)}{\Gamma(a+3/2-n)} \end{pmatrix} \begin{pmatrix} z^{a-n+1} K_1(z) \\ \mathrm{i} z^{a-n+1} K_0(z) \end{pmatrix} \tag{A.9}$$

or

$$\Phi^n_\lambda \approx 2^n (2\lambda+2n+1)^{\lambda+1} \begin{pmatrix} \frac{\Gamma(\lambda+n+1/2)}{\Gamma(\lambda+1/2)} & 0 \\ 0 & \frac{\Gamma(\lambda+n+3/2)}{\Gamma(\lambda+3/2)} \end{pmatrix} \begin{pmatrix} z^{\lambda+1} K_1(z) \\ \mathrm{i} z^{\lambda+1} K_0(z) \end{pmatrix}. \tag{A.10}$$

From here, taking into account (A.7), it is easy to conclude that for any $\lambda > 0$ the states Φ^n_λ are well behaved near the origin.

A.2. Behaviour at the infinity

For big values of the variables z or r, we have that

$$\Phi^n_{a-n} \approx G^-_{a-n} G^-_{a-n+1} \cdots G^-_{a-2} G^-_{a-1} \Phi^0_a \tag{A.11}$$

and taking into account that in this region we can approximate

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z} \tag{A.12}$$

after a simple calculation we get

$$\Phi^n_{a-n} \approx \frac{(-\sigma_2)^n}{(2a+1)^n} \frac{\Gamma(a-n+1/2)\Gamma(2a+1)}{\Gamma(a+1/2)\Gamma(2a-n+1)} \Phi^0_a \tag{A.13}$$

or

$$\Phi^n_\lambda \approx \frac{(-\sigma_2)^n}{(2\lambda+2n+1)^n} \frac{\Gamma(\lambda+1/2)\Gamma(2\lambda+2n+1)}{\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n+1)} \Phi^0_{\lambda+n}. \tag{A.14}$$

Using (A.12), we conclude that for any $\lambda > 0$ the states Φ^n_λ are well behaved at infinity.

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